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# CATALAN COMBINATORICS

## DOCENTSHIP LECTURE

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# THE HISTORY OF THE CATALAN SEQUENCE

1,1,2,5,14,42,132,429,1430,4862,...

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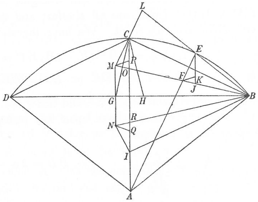


Minggatu (1692–1763)

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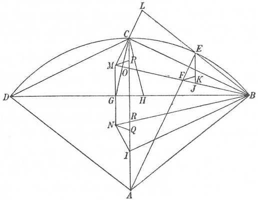


$$\sin(2x) = 2\sin(x) - \sum_{n=1}^{\infty} C_n \frac{(\sin(x))^{2n+1}}{4^{n-1}}$$

$$\sin(4x) = 4\sin(x) - 10(\sin(x))^3 + \sum_{n=1}^{\infty} (16C_n - 2C_{n+1}) \frac{(\sin(x))^{2n+3}}{4^n}$$

Minggatu (1692–1763)

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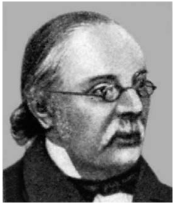
# A DEFINITION OF THE CATALAN SEQUENCE

**Definition**

The Catalan sequence is defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$



Eugène Catalan (1814–1894)

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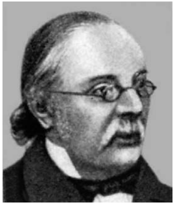

Eugène Catalan (1814–1894)

**Philosophical aspects:**

- What is the role of a proof?
- What do you (not) like about a certain proof?
- What are the strengths of a certain proof?

# Basic properties

## Definition

The Catalan sequence is defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

## Lemma

*The following statements hold (for $n \geq 1$):*

(i) $C_n = \dfrac{4n-2}{n+1} C_{n-1}$,

(ii) $C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}$.

# Proof by induction and direct computation

$$C_n = \frac{4n-2}{n+1} C_{n-1}$$

**Proof.**

Induction step:

$$\frac{4n-2}{n+1} C_{n-1}$$

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# Proof by induction and direct computation

$$C_n = \frac{4n-2}{n+1} C_{n-1}$$

**Proof.**

Induction step:

$$\frac{4n-2}{n+1} C_{n-1} = \frac{2(2n-1)}{n+1} \cdot \frac{1}{n} \binom{2(n-1)}{n-1}$$

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□

# Proof by hidden idea

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}$$

**Proof.**

Define

$$a(n,j) = \frac{2j-n}{2n(n+1)} \binom{2j}{j} \binom{2(n-j)}{(n-j)}.$$

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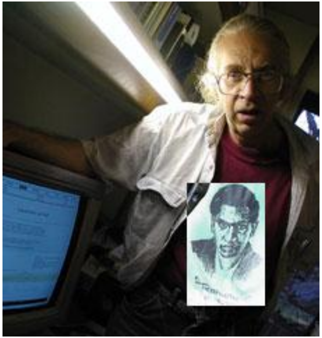
Then, a direct computation shows $a(n,i+1) - a(n,i) = C_i \cdot C_{n-1-i}$. Thus,

$$\sum_{i=0}^{n-1} C_i \cdot C_{n-1-i} = \sum_{i=0}^{n-1} (a(n,i+1) - a(n,i)) = a(n,n) - a(n,0)$$

$$= \frac{n}{2n(n+1)} \binom{2n}{n} - \frac{-n}{2n(n+1)} \binom{2n}{n} = C_n.$$

□

# Gosper’s algorithm



Bill Gosper (1943–)

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{99^2} \cdot \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \cdot \frac{26390n + 1103}{396^{4n}}$$

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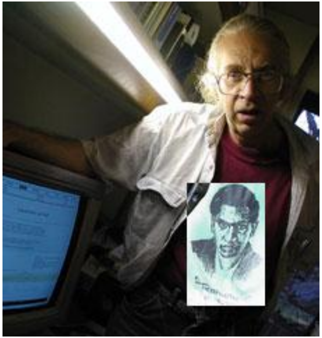

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**Problem**

Given a sequence $z_i$, find $a_i$ with

$$z_i = a_{i+1} - a_i.$$

Then,

$$\sum_{i=0}^{n-1} z_i = a_n - a_0.$$

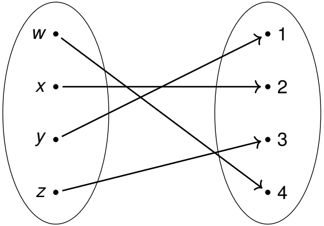
# Analysis of elementary algebraic proofs

- Understandable with basic mathematical knowledge
- Very little room for error
- Easy to implement in a computer
- Hard to remember
- Hard to communicate

# Bijective proofs

## Strategy

To prove $a = b$ for $a, b \in \mathbb{N}$, construct sets $A$ and $B$ with $|A| = a$ and $|B| = b$ and a bijective function $f\colon A \to B$.




## Definition

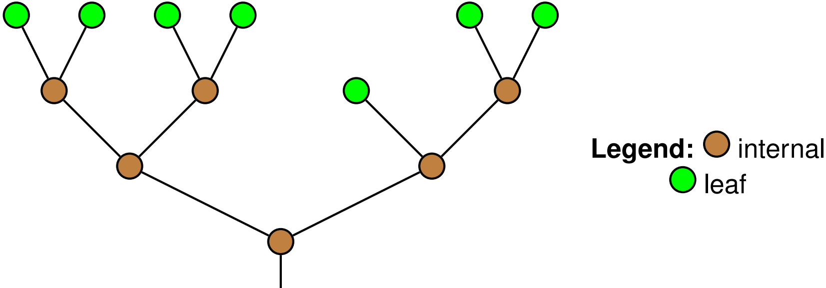
A function $f\colon A \to B$ is bijective if there is a function $g\colon B \to A$ such that $g(f(a)) = a$ and $f(g(b)) = b$ for all $a \in A$ and $b \in B$.

# Binary trees

## Definition

A (full) binary tree is either:

- A single vertex.
- A tree whose root node has (exactly) two subtrees each of which is a (full) binary tree



# Number of binary trees

**Lemma**

*The number of binary trees with n internal vertices is given by $C_n$.*

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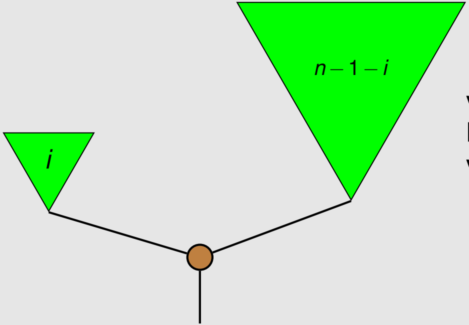


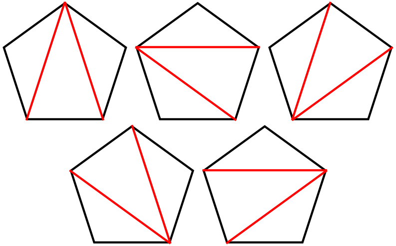

Put a binary with $i$ internal vertices on the left tree, thus a binary tree with $n - 1 - i$ vertices on the right tree.

$$\rightsquigarrow C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i} \qquad \square$$

# Triangulations of convex polygons

**Definition**

A triangulation of a polygon is a subdivision into triangles.

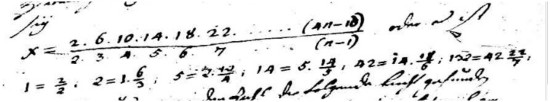


Leonhard Euler (1707–1783)

# Number of triangulations

**Lemma**

*The number of triangulations of a convex $(n+2)$-gon is $C_n$.*



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Taking the dual graph provides a bijection to binary trees:

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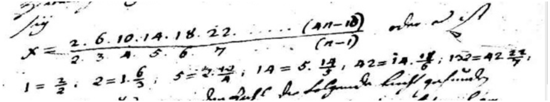
$$x = \frac{2.6.10.14.18.22.\ldots(4n-10)}{2.3.4.5.6.7.\ldots(n-1)}$$

$1 = \frac{2}{2};\ 2 = 1.\frac{6}{3};\ 5 = 2.\frac{10}{4};\ 14 = 5.\frac{14}{5},\ 42 = 14.\frac{18}{6};\ 132 = 42.\frac{22}{7};$

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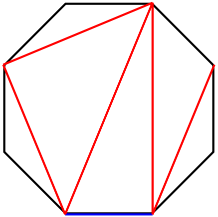

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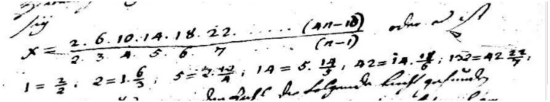

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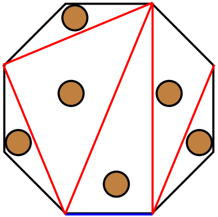

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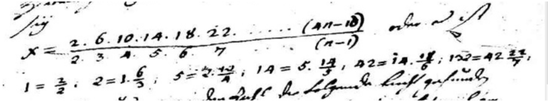

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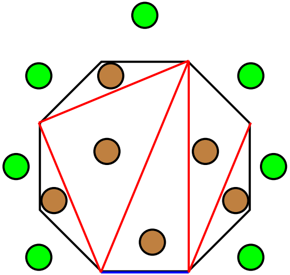

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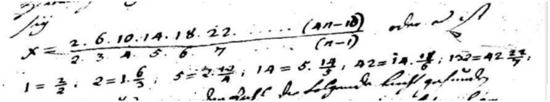

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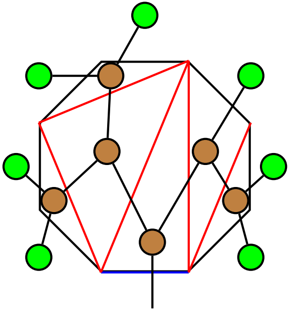

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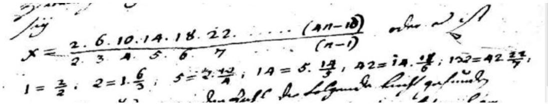
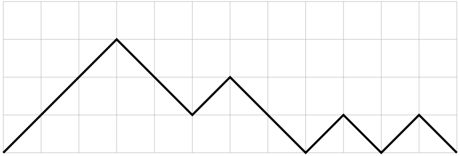


# Dyck paths

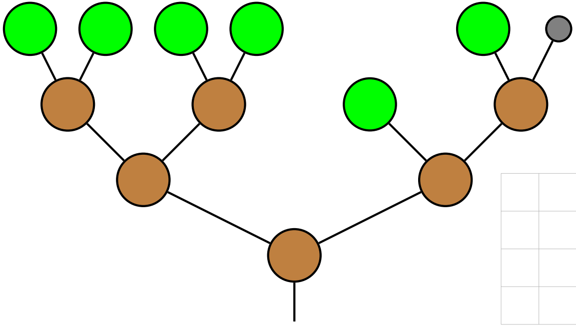
**Definition**

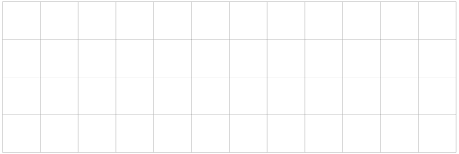
A Dyck path is a path from $(0,0)$ to $(2n,0)$ taking only steps $(1,1)$ and $(1,-1)$, whose $y$-coordinate is always nonnegative.



Walther von Dyck (1856–1930)

# Number of Dyck paths is $C_n$





**Preorder traversal:**

- root,
- then left tree,
- then right tree

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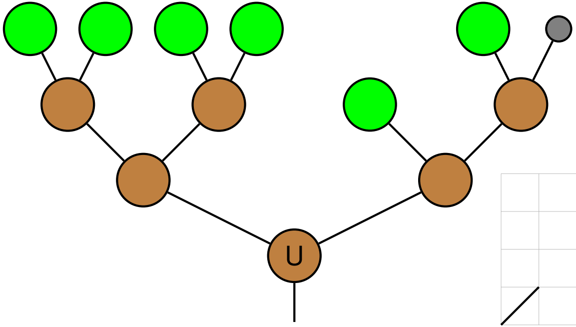


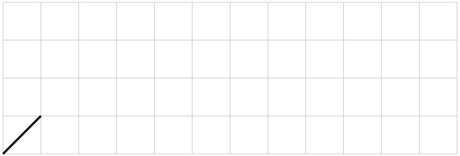

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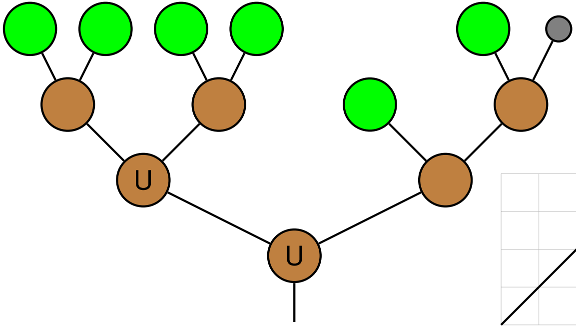


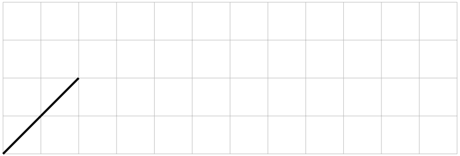

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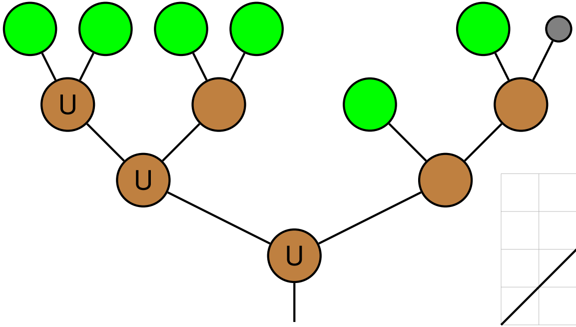


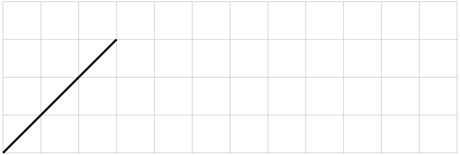

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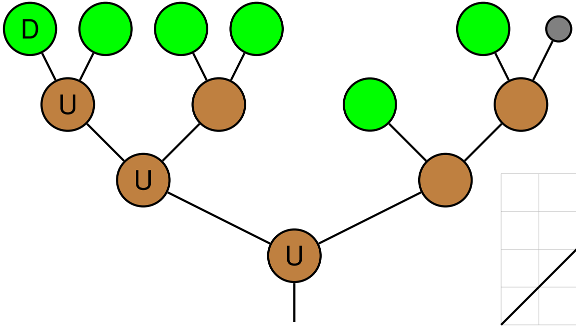


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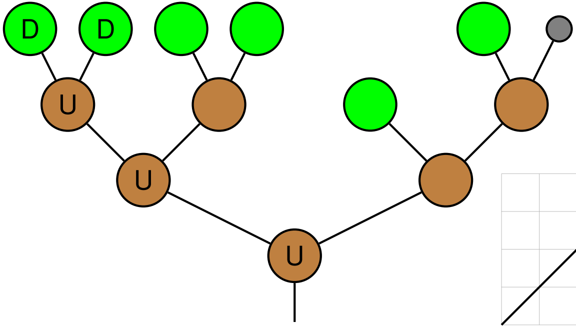


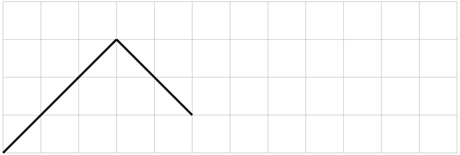

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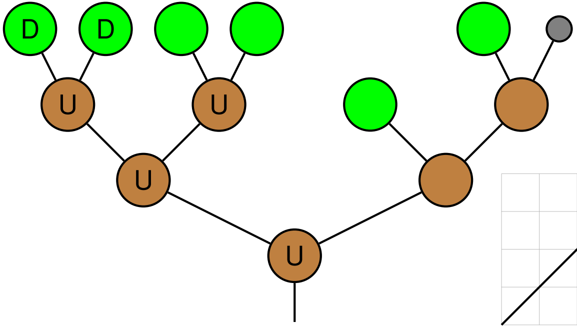


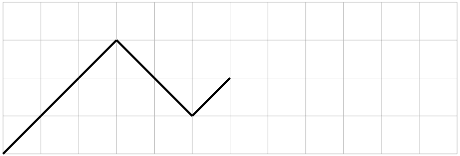

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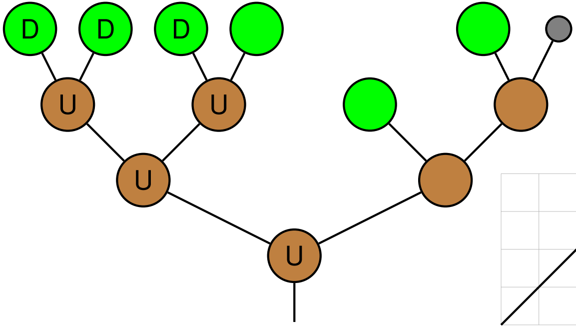


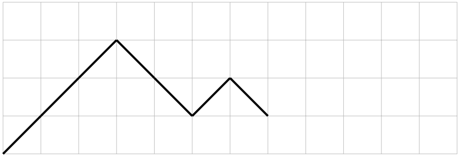

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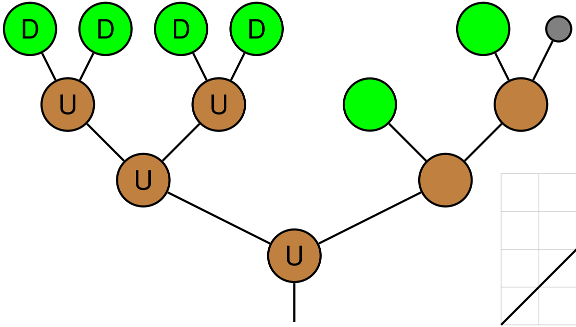


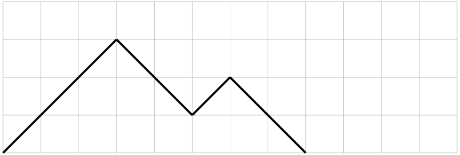

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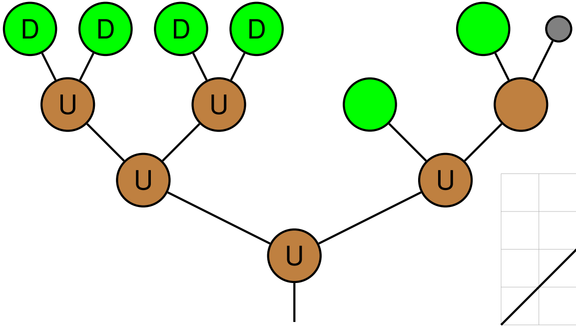


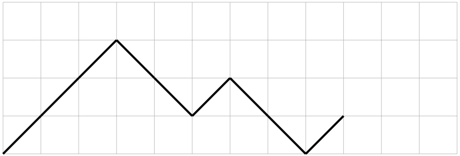

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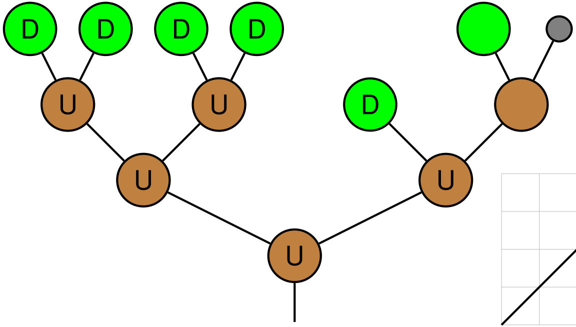


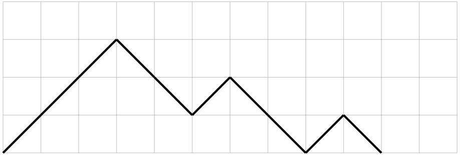

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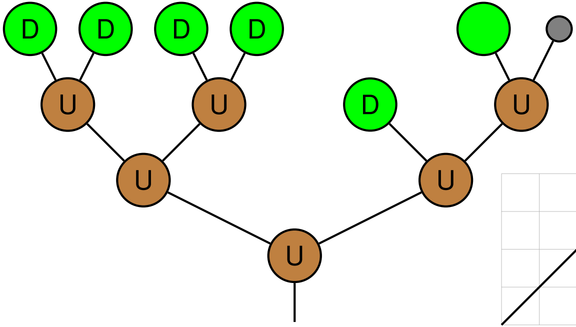


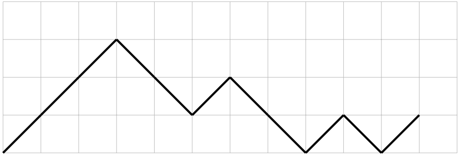

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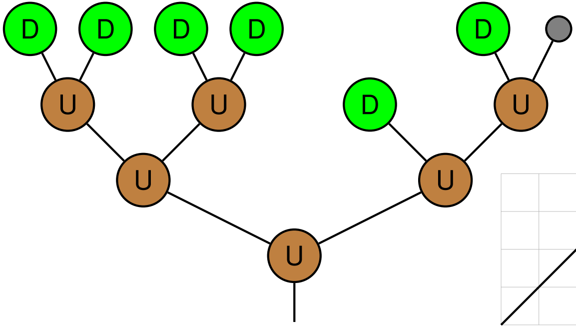


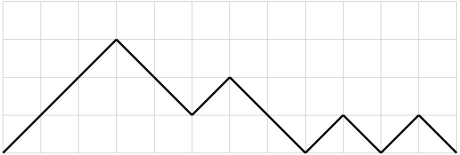

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# Bijective proof $2(2n-1)C_{n-1} = (n+1)C_n$

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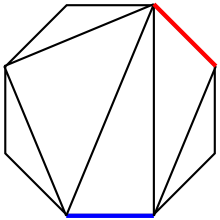
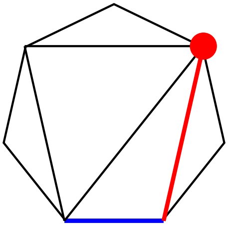
Collapse triangle with marked boundary edge:

$$f: \{\text{ triangulations with marked boundary edge }\}$$
$$\downarrow$$
$$\{\text{triangulations with oriented marked edge (boundary or diagonal)}\}$$





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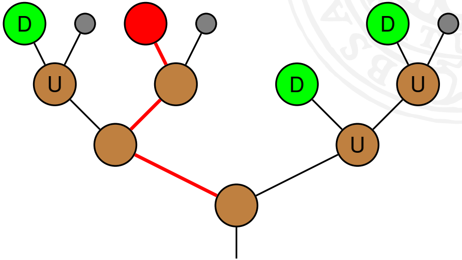

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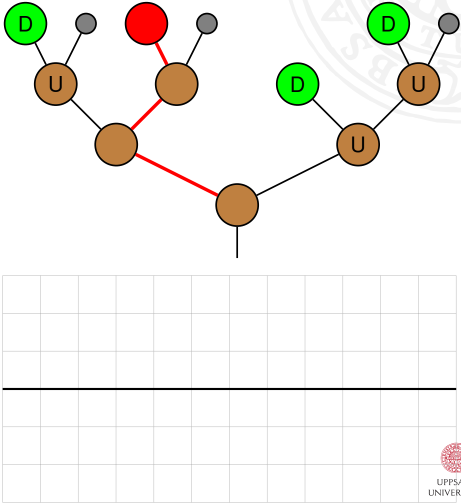

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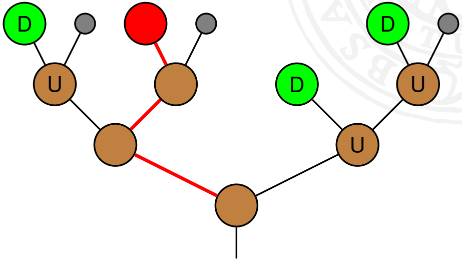


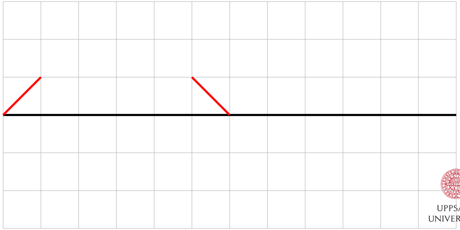

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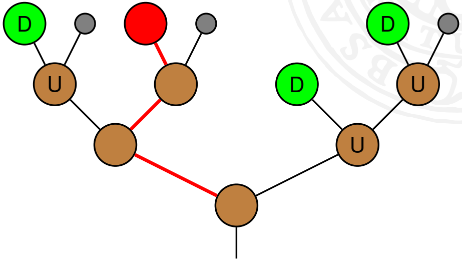


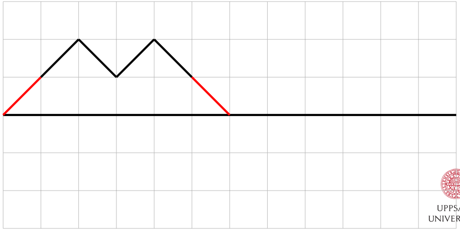

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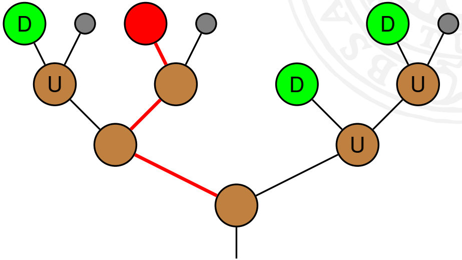


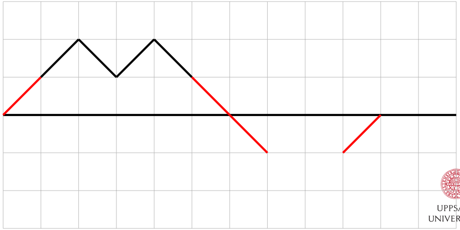

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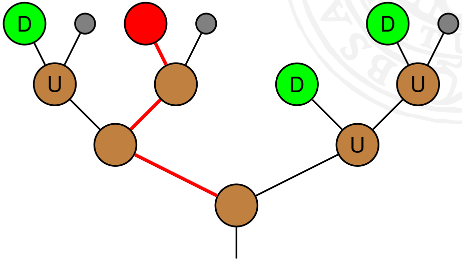


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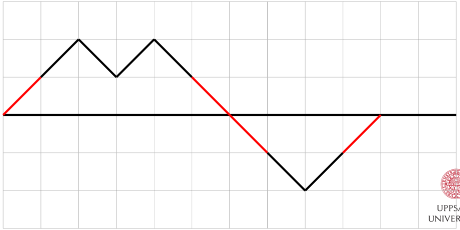

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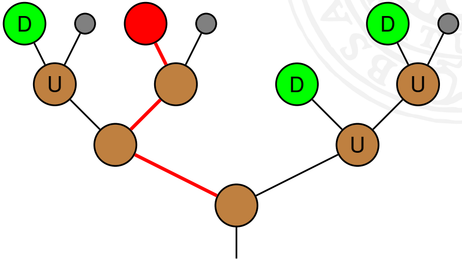
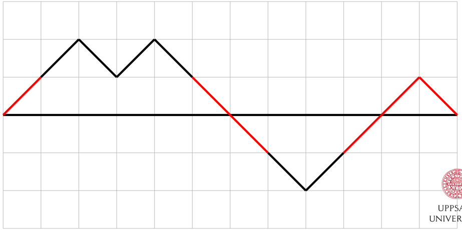


# Analysis of bijective proofs

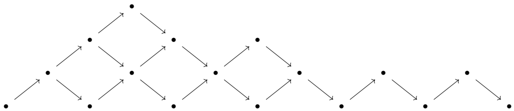
- Much more pleasing to the eye
- Easy to communicate
- Easy to remember
- Requires more preknowledge
- More room for error
- Harder to reduce to the axioms

# Outlook: Research perspectives on Catalan numbers

- Nakayama algebras
- Tilting modules
- $A_\infty$-algebras and super-Catalan numbers

# Nakayama algebras

There is a bijection between (admissible) quotients of the ring of upper triangular $(n+1) \times (n+1)$-matrices and Dyck paths.



**Theorem (Chavli–Marczinzik '22)**

*The number of projective modules of injective dimension one for the Nakayama algebra corresponding to a Dyck path is equal to the number of fixed points of the* $321$*-avoiding permutation corresponding to it under the Billey–Jockusch–Stanley bijection.*

# Tilting modules

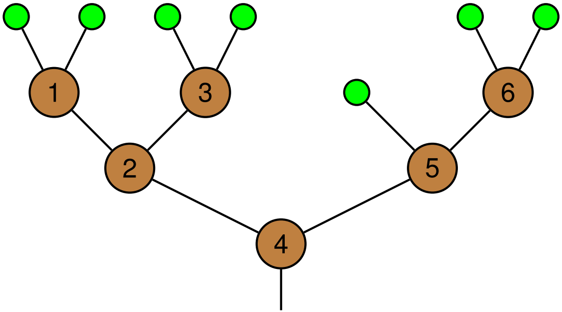
**Theorem (Flores, Kimura, Rognerud '20)**

*There are bijections between:*

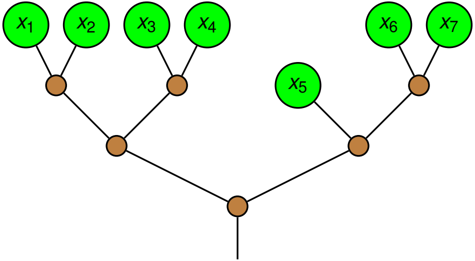
(1) *Binary trees with $n$ internal vertices,*

(2) *Minimal adapted partial orders on* $\{1, 2, \ldots, n\}$*,*

(3) *Tilting modules for upper triangular* $n \times n$*-matrices.*

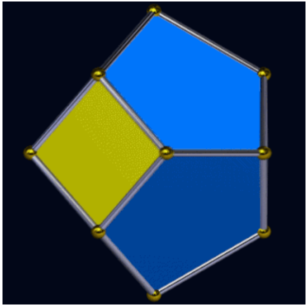


# $A_\infty$-algebras




Associativity: Result independent of binary tree.

Multiplications with several inputs, i.e. non-binary trees and corresponding multiplication structures.
$\rightsquigarrow$ super Catalan numbers
$1, 1, 3, 11, 45, 197, \ldots$



# Want to learn more?

I recommend lectures by:

Alissa S. Crans

Xavier Viennot